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Covariant quantization of the superparticle using pure spinors

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Abstract: The ten-dimensional superparticle is covariantly quantized by constructing a BRST operator from the fermionic Green-Schwarz constraints and a bosonic pure spinor variable. This same method was recently used for covariantly quantizing the superstring, and it is hoped that the simpler case of the superparticle will be useful for those who want to study this quantization method. It is interesting that quantization of the superparticle action closely resembles quantization of the worldline action for Chern-Simons theory.

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Contents

1. Introduction **1**

2. Review of standard superparticle description **2**

3. Worldline description of Chern-Simons theory **4**
- 3.1 Worldline action 4
- 3.2 Non-abelian Chern-Simons theory 5

4. Covariant description of Super-Yang-Mills theory **6**

5. Covariant quantization of the superparticle **8**
- 5.1 Pure spinors 8
- 5.2 Quantization 10
- 5.3 Non-abelian super-Yang-Mills background 12

A. Superparticle cohomology at zero momentum **13**
- A.1 Equivalence of Q and $\widehat{Q}$ cohomologies 14
- A.2 Evaluation of $\widehat{Q}$ cohomology 15

1. Introduction

Recently, the ten-dimensional superstring was covariantly quantized by constructing a BRST operator from the fermionic Green-Schwarz constraints and a bosonic pure spinor variable [1]. Although this method was successfully used for computing tree-level scattering amplitudes, the construction of the BRST operator is non conventional so it is a bit mysterious why this method works. This same BRST construction can be used for covariantly quantizing the ten-dimensional superparticle and it is hoped that by studying this simpler model, some of the mysteries will be easier to understand.

Since the spectrum of the ten-dimensional superparticle contains a spin-one field, the constraints of the worldline action should imply spacetime gauge invariances as well as spacetime equations of motion. This differs from the worldline actions for the particle or spinning particle where the constraints imply the Klein-Gordon or Dirac equations of motion, but do not imply spacetime gauge invariances. One worldline action which does describe a theory with spacetime gauge invariance is the worldline version of Witten's action for Chern-Simons theory [2]. It will turn out that the constraints and quantization of this Chern-Simons action closely resemble the constraints and quantization of the pure spinor version of the superparticle action.[1]

Section 2 of this paper will review the problems with quantizing the standard superparticle action. In section 3, the worldline action for Chern-Simons will be discussed. Section 4 will review the superspace description of ten-dimensional super-Yang-Mills. And in section 5, the pure spinor version of the superparticle action will be quantized in a manner similar to the Chern-Simons action. The appendix will contain a computation of the zero-momentum BRST cohomology of the superparticle.

2. Review of standard superparticle description

The standard action for the ten-dimensional superparticle is [3]

$$S = \int d\tau \left(\Pi^m P_m + e P^m P_m \right) , \tag{2.1}$$

where

$$\Pi^m = \dot{x}^m - \frac{i}{2} \dot{\theta}^\alpha \gamma^m_{\alpha\beta} \theta^\beta , \tag{2.2}$$

P_m is the canonical momentum for x^m, and e is the Lagrange multiplier which enforces the mass-shell condition. The gamma matrices $\gamma^m_{\alpha\beta}$ and $\gamma_m^{\alpha\beta}$ are 16×16 symmetric matrices which satisfy $\gamma^{(m}_{\alpha\beta} \gamma^{n)\ \beta\gamma} = 2\eta^{mn} \delta^\gamma_\alpha$. In the Weyl representation, $\gamma^m_{\alpha\beta}$ and $\gamma_m^{\alpha\beta}$ are the off-diagonal blocks of the 32×32 Γ^m matrices.

The action of (2.1) is spacetime-supersymmetric under

$$\delta\theta^\alpha = \epsilon^\alpha , \qquad x^m = \frac{i}{2} \theta \gamma^m \epsilon , \qquad \delta P_m = \delta e = 0 ,$$

and is also invariant under the local κ transformations [4]

$$\delta\theta^\alpha = P^m (\gamma_m \kappa)^\alpha , \qquad \delta x^m = -\frac{i}{2} \theta \gamma^m \delta\theta , \qquad \delta P_m = 0, \quad \delta e = i \dot{\theta}^\beta \kappa_\beta . \tag{2.3}$$

The canonical momentum to θ^α, which will be called p_α, satisfies

$$p_\alpha = \frac{\delta L}{\delta \dot{\theta}^\alpha} = -\frac{i}{2} P^m (\gamma_m \theta)_\alpha ,$$

so canonical quantization requires that physical states are annihilated by the fermionic Dirac constraints defined by

$$d_\alpha = p_\alpha + \frac{i}{2} P_m (\gamma^m \theta)_\alpha . \tag{2.4}$$

[1] The similarity of the two worldline actions was first pointed out to me by Warren Siegel.

Since $\{p_\alpha, \theta^\beta\} = -i\delta_\alpha^\beta$, these constraints satisfy the Poisson brackets

$$\{d_\alpha, d_\beta\} = P_m \gamma^m_{\alpha\beta} \,, \tag{2.5}$$

and since $P^m P_m = 0$ is also a constraint, eight of the sixteen Dirac constraints are first-class and eight are second-class. One can easily check that the eight first-class Dirac constraints generate the κ transformations of (2.3), however, there is no simple way to covariantly separate out the second-class constraints.

Nevertheless, one can easily quantize the superparticle in a non-Lorentz covariant manner and obtain the physical spectrum. Assuming non-zero P^+, the local fermionic κ-transformations can be used to gauge-fix $(\gamma^+\theta)_\alpha = 0$ where $\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^9)$. In this gauge, the action of (2.1) simplifies to the quadratic action [5]

$$\begin{aligned} S &= \int d\tau \left(\dot{x}^m P_m + \frac{i}{2} P^+ (\dot\theta \gamma^- \theta) + e P^m P_m \right) \\ &= \int d\tau \left(\dot{x}^m P_m + \frac{i}{4} \dot\sigma_a \sigma_a + e P^m P_m \right) \,, \end{aligned} \tag{2.6}$$

where $\sigma_a = \sqrt{2P^+}(\gamma^-\theta)_a$ and $a = 1$ to 8 is an SO(8) chiral spinor index.

Canonical quantization of (2.6) implies that $\{\sigma_a, \sigma_b\} = 2\delta_{ab}$. So σ_a acts like a "spinor" version of the SO(8) Pauli matrices $\sigma^j_{a\dot b}$ which satisfy

$$\sigma^j_{a\dot c} \sigma^j_{b\dot d} + \sigma^j_{b\dot c} \sigma^j_{a\dot d} = 2\delta_{ab}\delta_{\dot c\dot d} \,,$$

where j and $\dot b$ are SO(8) vector and antichiral spinor indices. One can therefore define the quantum-mechanical wavefunction $\Psi(x)$ to carry either an SO(8) vector index, $\Psi_j(x)$, or an SO(8) antichiral spinor index, $\Psi_{\dot a}(x)$, and the anticommutation relations of σ_a are reproduced by defining

$$\sigma^a \Psi_j(x) = \sigma_j^{a\dot b} \Psi_{\dot b}(x) \,, \qquad \sigma_a \Psi_{\dot b}(x) = \sigma^j_{a\dot b} \Psi_j(x) \,. \tag{2.7}$$

Furthermore, the constraint $P_m P^m$ implies the linearized equations of motion $\partial_m \partial^m \Psi_j = \partial_m \partial^m \Psi_{\dot b} = 0$.

So the physical states of the superparticle are described by a massless SO(8) vector $\Psi_j(x)$ and a massless SO(8) antichiral spinor $\Psi_{\dot a}(x)$ which are the physical states of $d = 10$ super-Yang-Mills theory. However, this description of super-Yang-Mills theory only manifestly preserves an SO(8) subgroup of the super-Poincaré group, and one would like a more covariant method for quantizing the theory. Covariant quantization can be extremely useful if one wants to compute more than just the physical spectrum in a flat background. For example, non-covariant methods are extremely clumsy for computing scattering amplitudes or for generalizing to curved backgrounds.

Since the super-Yang-Mills spectrum contains a massless vector, one expects the covariant superparticle constraints to generate the spacetime gauge invariances of this vector. Note that these constraints are not present in the gauge-fixed action of (2.6) since Ψ_j describes only the transverse degrees of freedom of the $SO(9,1)$ vector. Before describing the covariant constraints which generate the gauge invariances of this vector, it will be useful to first review the worldline action for Chern-Simons theory which also has constraints related to spacetime gauge invariances.

3. Worldline description of Chern-Simons theory

Since the gauge invariance of a massless vector field is $\delta A_\mu = \partial_\mu \Lambda$, one might guess that the worldline action for such a field should contain the constraints P_μ. Although these constraints are too strong for describing Yang-Mills theory, they are just right for describing $d = 3$ Chern-Simons theory where the field-strength of A_μ vanishes on-shell.

3.1 Worldline action

As was shown in [2], Chern-Simons theory can be described using the worldline action

$$S = \int d\tau (\dot{x}^\mu P_\mu + l^\mu P_\mu)\,, \tag{3.1}$$

where $\mu = 0$ to 2 and l^μ are Lagrange multipliers for the constraints. Since the constraints are first-class, the action can be quantized using the BRST method. After gauging $l^\mu = -\frac{1}{2}P^\mu$, the gauge-fixed action is

$$S = \int d\tau \left(\dot{x}^\mu P_\mu - \frac{1}{2} P^\mu P_\mu + \dot{c}^\mu b_\mu \right) \tag{3.2}$$

with the BRST operator

$$Q = c^\mu P_\mu\,, \tag{3.3}$$

where (c^μ, b_μ) are fermionic Fadeev-Popov ghosts and anti-ghosts.

To show that the cohomology of the BRST operator describes Chern-Simons theory, note that the most general wavefunction constructed from a ground state annihilated by b^μ is

$$\Psi(c,x) = C(x) + c^\mu A_\mu(x) + \frac{i}{2}\epsilon_{\mu\nu\rho} c^\mu c^\nu A^{*\rho}(x) + \frac{i}{6}\epsilon_{\mu\nu\rho} c^\mu c^\nu c^\rho C^*(x)\,, \tag{3.4}$$

where the expansion in c^μ terminates since c^μ is fermionic. One can check that

$$Q\Psi = -ic^\mu \partial_\mu C - \frac{i}{2} c^\mu c^\nu \partial_{[\mu} A_{\nu]} + \frac{1}{6}\epsilon_{\mu\nu\rho} c^\mu c^\nu c^\rho \partial_\sigma A^{*\sigma}(x)\,. \tag{3.5}$$

JHEP09(2001)016

So $Q\Psi = 0$ implies that $A_\mu(x)$ satisfies the equations of motion $\partial_{[\mu}A_{\nu]} = 0$ which is the linearized equation of motion of the Chern-Simons field. Furthermore, if one defines the gauge parameter $\Omega(c,x) = i\Lambda(x) - c^\mu \omega_\mu(x) + \cdots$, the gauge transformation $\delta\Psi = Q\Omega$ implies $\delta A_\mu = \partial_\mu \Lambda$ which is the linearized gauge transformation of the Chern-Simons field. Similarly, $Q\Psi = 0$ and $\delta\Psi = Q\Omega$ implies that $A^{*\rho}$ satisfies the equation of motion $\partial_\sigma A^{*\sigma} = 0$ with the gauge invariance $\delta A^{*\sigma} = \epsilon^{\sigma\mu\nu}\partial_\mu w_\nu$, which are the linearized equations of motion and gauge invariance of the Chern-Simons antifield. The remaining fields, $C(x)$ and $C^*(x)$, describe the spacetime ghost and antighost of Chern-Simons theory.

These equations of motion and gauge invariances can be obtained from the Batalin-Vilkovisky version [6] of the abelian Chern-Simons spacetime action

$$\mathcal{S} = \int d^3x \left(\frac{1}{2}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + i A^{*\mu}\partial_\mu C \right), \tag{3.6}$$

where, in addition to the usual Chern-Simons action for A_μ, there is a term coupling the antifield $A^{*\mu}$ to the gauge variation of A_μ. The action of (3.6) can be written compactly in terms of the wavefunction Ψ of (3.4) as

$$\mathcal{S} = \frac{1}{2}\int d^3x \langle \Psi Q \Psi \rangle \,, \tag{3.7}$$

where $\langle\ \ \rangle$ is normalized such that $\langle c^\mu c^\nu c^\rho \rangle = i\epsilon^{\mu\nu\rho}$.

3.2 Non-abelian Chern-Simons theory

Up to now, only abelian Chern-Simons theory has been discussed, but it is easy to generalize to the non-abelian case. For example, the Batalin-Vilkovisky version of the non-abelian Chern-Simons action is

$$\begin{aligned} \mathcal{S} = \mathrm{Tr} \int d^3x \bigg(\epsilon^{\mu\nu\rho} \left(\frac{1}{2} A_\mu \partial_\nu A_\rho + \frac{ig}{3} A_\mu A_\nu A_\rho \right) + \\ + i A^{*\mu}(\partial_\mu C + ig[A_\mu, C]) - gCCC^* \bigg) \,, \end{aligned} \tag{3.8}$$

which can be written compactly as

$$\mathcal{S} = \mathrm{Tr} \int d^3x \left\langle \frac{1}{2}\Psi Q \Psi + \frac{g}{3}\Psi\Psi\Psi \right\rangle, \tag{3.9}$$

where g is the Chern-Simons coupling constant and the fields in Ψ of (3.4) now carry Lie algebra indices.[2] Note that the non-linear equations of motion and gauge

[2] It is interesting to note that the Chern-Simons action can also be written in manifestly gauge-invariant notation as $\mathcal{S} = \frac{1}{2}\mathrm{Tr}\int_{\mathcal{M}} d^4x \langle (Q\Psi + g\Psi\Psi)^2 \rangle_4$ where $\mathcal{M}$ is a four-dimensional volume with a three-dimensional boundary at $x_3 = 0$, $Q = c^M P_M$ for $M = 0$ to 3, $\langle c^M c^N c^P c^Q \rangle_4 = \epsilon^{MNPQ}$, and Ψ now depends on x^3 and c^3. Using cyclicity of the trace, one finds $\mathcal{S} = \mathrm{Tr}\int_{\mathcal{M}} d^4x \langle Q(\frac{1}{2}\Psi Q\Psi + \frac{g}{3}\Psi\Psi\Psi)\rangle_4 = \mathrm{Tr}\int d^4x \langle \delta(c^3)\delta(x^3)(\frac{1}{2}\Psi Q \Psi + \frac{g}{3}\Psi\Psi\Psi)\rangle_4$ since $\int_{\mathcal{M}} d^4x \langle Q\Lambda\rangle_4$ only gets contributions from the three-dimensional boundary at $x_3 = 0$. So $\mathcal{S}$ coincides with the Chern-Simons action defined in (3.9).

invariances associated with this action are

$$Q\Psi + g\Psi\Psi = 0\,, \qquad \delta\Psi = Q\Omega + g[\Omega, \Psi]\,. \tag{3.10}$$

To construct a wordline action for non-abelian Chern-Simons theory with SO(N) gauge group, one introduces N real fermionic variables η_I for $I = 1$ to N and modifies the worldline action of (3.2) to[3]

$$S = \int d\tau \left(\dot{x}^\mu P_\mu - \frac{1}{2} P^\mu P_\mu - \frac{i}{2} \eta_I \nabla \eta_I + \dot{c}^\mu b_\mu \right), \tag{3.11}$$

where $\nabla\eta_I = \dot{\eta}_I + g\eta_J \dot{x}^\mu \tilde{A}^{IJ}_\mu(x)$ and $\tilde{A}^{IJ}_\mu(x) = -\tilde{A}^{JI}_\mu(x)$ is a non-abelian Chern-Simons background field [7]. Note that the constraints P_μ are conserved when the background is on-shell since

$$\begin{aligned}
\dot{P}_\mu &= \frac{ig}{2} \left(\frac{\partial}{\partial\tau} \left(\eta_I \eta_J \tilde{A}^{IJ}_\mu \right) - \eta_I \eta_J \dot{x}^\nu \partial_\mu \tilde{A}^{IJ}_\nu \right) \\
&= -\frac{ig}{2} \eta_I \eta_J \dot{x}^\nu \left(\partial_{[\mu} \tilde{A}^{IJ}_{\nu]} + g \tilde{A}^{IK}_{[\mu} \tilde{A}^{KJ}_{\nu]} \right) = 0\,.
\end{aligned} \tag{3.12}$$

After gauge-fixing, the BRST charge is still $Q = c^\mu P_\mu$, but because of the background gauge field $\tilde{A}^{IJ}_\mu$ in (3.11), the canonical momentum for x^μ is now $\partial L/\partial \dot{x}^\mu = P_\mu - \frac{ig}{2} \eta_I \eta_J \tilde{A}^{IJ}_\mu$. Since $\{\eta_I, \eta_J\} = \delta_{IJ}$,

$$\begin{aligned}
Q \left(\eta_I \eta_J \Psi^{IJ} \right) = \eta_I \eta_J \Bigg(&-ic^\mu (\nabla_\mu C)^{IJ} - \frac{i}{2} c^\mu c^\nu \left(\nabla_{[\mu} A_{\nu]} \right)^{IJ} + \\
&+ \frac{1}{6} \epsilon_{\mu\nu\rho} c^\mu c^\nu c^\rho (\nabla_\sigma A^{*\sigma})^{IJ} \Bigg),
\end{aligned} \tag{3.13}$$

where $(\nabla_\mu s)^{IJ} = \partial_\mu s^{IJ} + g(s^{IK} \tilde{A}^{KJ}_\mu - s^{JK} \tilde{A}^{KI}_\mu)$ and Ψ^{IJ} is defined as in (3.4). So in a background gauge field $\tilde{A}^{IJ}_\mu$, the linearized equations of motion of the non-abelian Chern-Simons field and antifield are correctly described by $Q(\eta_I \eta_J \Psi^{IJ}) = 0$.

Using intuition learned from this worldline description of Chern-Simons theory, it will be shown how to quantize the superparticle in a similar manner. However, before performing this quantization, it will be useful to first review the superspace description of ten-dimensional super-Yang-Mills theory.

4. Covariant description of Super-Yang-Mills theory

Although on-shell super-Yang-Mills theory can be described by the SO(8) wavefunctions $\Psi_j(x)$ and $\Psi_{\dot{a}}(x)$ of (2.7) satisfying the linearized equations of motion $\partial_m \partial^m \Psi_j = \partial_m \partial^m \Psi_{\dot{a}} = 0$, there are more covariant descriptions of the theory. Of

[3] For U(N) gauge group, one introduces N complex fermionic variables $(\eta_I, \overline{\eta}^I)$ with the action $-i \int d\tau\, \overline{\eta}^I \nabla \eta_I$.

course, there is a Poincaré-covariant description using an SO(9,1) vector field $a_m(x)$ and an SO(9,1) spinor field $\chi^\alpha(x)$ transforming in the adjoint representation of the gauge group which satisfy the equations of motion

$$\partial^m f_{mn} + ig[a^m, f_{mn}] = 0\,, \qquad \gamma^m_{\alpha\beta}\left(\partial_m \chi^\beta + ig\left[a_m, \chi^\beta\right]\right) = 0\,, \tag{4.1}$$

and gauge invariance

$$\delta a_m = \partial_m s + ig[a_m, s]\,, \qquad \delta\chi^\alpha = ig[\chi^\alpha, s]\,, \qquad \delta f_{mn} = ig[f_{mn}, s]\,, \tag{4.2}$$

where $f_{mn} = \partial_{[m} a_{n]} + ig[a_m, a_n]$ is the Yang-Mills field strength and g is the super-Yang-Mills coupling constant. However, there is also a super-Poincaré covariant description using an SO(9,1) spinor wavefunction $A_\alpha(x,\theta)$ defined in $d = 10$ superspace. As will be explained below, on-shell super-Yang-Mills theory can be described by a spinor superfield $A_\alpha(x,\theta)$ transforming in the adjoint representation which satisfies the superspace equation of motion [8]

$$\gamma^{\alpha\beta}_{mnpqr}(D_\alpha A_\beta + igA_\alpha A_\beta) = 0 \tag{4.3}$$

for any five-form direction $mnpqr$, with the gauge invariance

$$\delta A_\alpha = D_\alpha \Lambda + ig[A_\alpha, \Lambda]\,, \tag{4.4}$$

where $\Lambda(x,\theta)$ is any scalar superfield and

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}(\gamma^m\theta)_\alpha \partial_m$$

is the supersymmetric derivative.

One can also define field strengths constructed from A_α by

$$\begin{aligned} B_m &= -\frac{i}{8}\gamma^{\alpha\beta}_m(D_\alpha A_\beta + igA_\alpha A_\beta)\,, \\ W^\alpha &= \frac{1}{10}\gamma^{\alpha\beta}_m(D_\alpha B^m - \partial^m A_\alpha + ig[A_\alpha, B^m])\,, \\ F_{mn} &= \partial_{[m}B_{n]} + ig[B_m, B_n] = \frac{1}{8}(\gamma_{mn})_\alpha{}^\beta(D_\beta W^\alpha + ig\{A_\beta, W^\alpha\})\,. \end{aligned} \tag{4.5}$$

Under the gauge transformation of (4.4),

$$\delta B_m = \partial_m \Lambda + ig[B_m, \Lambda]\,, \qquad \delta W^\alpha = ig[W^\alpha, \Lambda]\,, \qquad \delta F^{mn} = ig[F^{mn}, \Lambda]\,. \tag{4.6}$$

To show that $A_\alpha(x,\theta)$ describes on-shell super-Yang-Mills theory, it will be useful to first note that in ten dimensions any symmetric bispinor $f_{\alpha\beta}$ can be decomposed in terms of a vector and a five-form as $f_{\alpha\beta} = \gamma^m_{\alpha\beta} f_m + \gamma^{mnpqr}_{\alpha\beta} f_{mnpqr}$ and any antisymmetric bispinor $f_{\alpha\beta}$ can be decomposed in terms of a three-form as $f_{\alpha\beta} = \gamma^{mnp}_{\alpha\beta} f_{mnp}$. Since $\{D_\alpha, D_\beta\} = i\gamma^m_{\alpha\beta}\partial_m$, one can check that $\delta A_\alpha = D_\alpha\Lambda + ig[A_\alpha, \Lambda]$ is indeed a gauge invariance of (4.3).

Using $\Lambda(x,\theta) = h_\alpha(x)\theta^\alpha + j_{\alpha\beta}(x)\theta^\alpha\theta^\beta$, one can gauge away $(A_\alpha(x))|_{\theta=0}$ and the three-form part of $(D_\alpha A_\beta(x))|_{\theta=0}$. Furthermore, eq. (4.3) implies that the five-form part of $(D_\alpha A_\beta(x))|_{\theta=0}$ vanishes. So the lowest non-vanishing component of $A_\alpha(x,\theta)$ in this gauge is the vector component $(D\gamma_m A(x))|_{\theta=0}$ which will be defined as $8ia_m(x)$. Continuing this type of argument to higher order in θ^α, one finds that there exists a gauge choice such that

$$A_\alpha(x,\theta) = \frac{i}{2}(\gamma^m\theta)_\alpha a_m(x) + \frac{i}{12}(\theta\gamma^{mnp}\theta)(\gamma_{mnp})_{\alpha\beta}\chi^\beta(x) + \cdots, \tag{4.7}$$

where $a_m(x)$ and $\chi^\beta(x)$ are SO$(9,1)$ vector and spinor fields satisfying (4.1) and where the component fields in $\ldots$ are functions of spacetime derivatives of $a_m(x)$ and $\chi^\beta(x)$. Furthermore, this gauge choice leaves the residual gauge transformations of (4.2) where $s(x) = (\Lambda(x))|_{\theta=0}$. Also, one can check that the $\theta = 0$ components of the superfields B_m, W^α and F_{mn} of (4.5) are a_m, χ^α and f_{mn}, respectively. So the equations of motion and gauge invariances of (4.3) and (4.4) correctly describe on-shell super-Yang-Mills theory.

One would now like to obtain this super-Poincaré covariant description of super-Yang-Mills theory by quantizing the superparticle. As will now be shown, this can be done by constructing a BRST-like operator out of the fermionic constraints d_α of (2.4).

5. Covariant quantization of the superparticle

In the case of Chern-Simons theory, the gauge transformation $\delta A_\mu = \partial_\mu\Lambda$ was generated by the constraints P_μ. So for the superparticle, the gauge transformation $\delta A_\alpha = D_\alpha\Lambda$ suggests using the constraints d_α. However, the constraints d_α are not all first-class, so

$$Q = \lambda^\alpha d_\alpha \tag{5.1}$$

would not be a nilpotent operator for generic λ^α. However, since (2.5) implies that $Q^2 = (\lambda^\alpha d_\alpha)^2 = \frac{1}{2}\lambda^\alpha\lambda^\beta\gamma^m_{\alpha\beta}P_m$, Q would be nilpotent if λ^α satisfied the condition

$$\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta = 0 \tag{5.2}$$

for $m = 0$ to 9. The condition of (5.2) is the definition of a pure spinor [9] and, as will now be shown, implies that only eleven components of λ^α are independent parameters.

5.1 Pure spinors

To solve the constraint of (5.2), it is convenient to first Wick-rotate to euclidean space and write the SO(10) spinor λ^α using SU(5) notation as $\lambda^{\pm\pm\pm\pm\pm}$ where $\pm$ denotes if the component is annihilated by $(\gamma_{2a-2} + i\gamma_{2a-1})$ or $(\gamma_{2a-2} - i\gamma_{2a-1})$ for $a = 1$ to 5.

JHEP09(2001)016

For example, the component λ^{+-+-+} is annihilated by $\gamma_0 + i\gamma_1$, $\gamma_2 - i\gamma_3$, $\gamma_4 + i\gamma_5$, $\gamma_6 - i\gamma_7$, and $\gamma_8 + i\gamma_9$. For a sixteen-component Weyl spinor, $\lambda^{\pm\pm\pm\pm\pm}$ contains either five $+$'s, three $+$'s, or one $+$, which transform, respectively under SU(5) as 1, $\overline{10}$, and 5 representations. These SU(5) representations will be called λ^+, λ_{ab} and λ^a where $a = 1$ to 5 and $\lambda_{ab} = -\lambda_{ba}$. One can check that any pure spinor λ^α satisfying (5.2) can be parameterized as

$$\lambda^+ = \gamma\,, \qquad \lambda_{ab} = \gamma u_{ab}\,, \qquad \lambda^a = -\frac{\gamma}{8}\epsilon^{abcde}u_{bc}u_{de}\,, \tag{5.3}$$

where $u_{ab} = -u_{ba}$ and γ are eleven independent parameters. To show that (5.3) satisfies (5.2), note that

$$\lambda\gamma^a\lambda = \lambda^+\lambda^a + \frac{1}{8}\epsilon^{abcde}\lambda_{bc}\lambda_{de}\,, \qquad \lambda\gamma_a\lambda = \lambda^b\lambda_{ab}\,, \tag{5.4}$$

where $\gamma^a = \frac{1}{\sqrt{2}}(\gamma_{2a-2} - i\gamma_{2a-1})$ and $\gamma_a = \frac{1}{\sqrt{2}}(\gamma_{2a-2} + i\gamma_{2a-1})$. Also note that the parameterization of (5.3) is singular when $\lambda^+ = 0$ since $\gamma \to 0$ and $u_{ab} \to \infty$ when $\lambda^+ \to 0$.

To obtain the BRST operator of (5.1) from gauge-fixing, start with the worldline action

$$S = \int d\tau \left(\dot{x}^m P_m - \frac{1}{2}P^m P_m + \dot{\theta}^\alpha p_\alpha + \frac{1}{2}\dot{u}_{ab}v^{ab} + l\left(\frac{\lambda^\alpha}{\gamma}\right)d_\alpha\right), \tag{5.5}$$

where p_α is the conjugate momentum for θ^α, v^{ab} is the conjugate momentum for u_{ab}, λ^α/γ is defined in terms of u_{ab} using the definition of (5.3), and l is the Lagrange multiplier for the constraint $(\lambda^\alpha/\gamma)d_\alpha$.[4] After gauge fixing $l = 0$, one obtains the action

$$S = \int d\tau \left(\dot{x}^m P_m - \frac{1}{2}P^m P_m + \dot{\theta}^\alpha p_\alpha + \frac{1}{2}\dot{u}_{ab}v^{ab} + \dot{\gamma}\beta\right) \tag{5.6}$$

with the BRST operator $Q = \lambda^\alpha d_\alpha$ where (γ, β) are the bosonic Fadeev-Popov ghost and antighost for the constraint $(\lambda^\alpha/\gamma)d_\alpha$.

Note that the action and BRST operator of (5.6) and (5.1) are spacetime supersymmetric since $\dot{x}^m P_m + \dot{\theta}^\alpha p_\alpha = \Pi^m P_m + \dot{\theta}^\alpha d_\alpha$ where Π^m and d_α are defined in (2.2) and (2.4). Furthermore, although (γ, β) and (u_{ab}, v^{ab}) do not transform linearly under Lorentz transformations, one can define Lorentz generators such that the pure spinor λ^α of (5.3) does transform linearly as $\delta\lambda^\alpha = \frac{1}{2}(\gamma^{mn})^\alpha{}_\beta\lambda^\beta$ under the transformation generated by N^{mn}. These Lorentz generators are given by

$$\begin{aligned} N^{ab} &= v^{ab}\,, \qquad N_{ab} = u_{ac}u_{bd}v^{cd} - u_{ab}\gamma\beta\,, \\ N_b^a &= u_{bc}v^{ac} - \frac{1}{5}\delta_b^a u_{cd}v^{cd}\,, \qquad N = \frac{5}{2}\gamma\beta - u_{ab}v^{ab}\,, \end{aligned} \tag{5.7}$$

[4]Note that the term $-\frac{1}{2}P^m P_m$ appears before gauge-fixing, implying that the action of (5.5) is not invariant under worldline reparameterizations. This fact is probably related to the indirect manner in which BRST invariance imposes the mass-shell condition.

where the 45 SO(10) Lorentz generators N^{mn} have been decomposed in terms of their irreducible SU(5) representations $(N^{ab}, N_{ab}, N^a_b, N)$ which transform as $(10, \overline{10}, 24, 1)$ representations. So the action and BRST operator of (5.6) and (5.1) are super-Poincaré invariant and it will now be shown that they correctly describe super-Yang-Mills theory.

5.2 Quantization

The most general super-Poincaré covariant wavefunction that can be constructed from $(x^m, \theta^\alpha, \lambda^\alpha)$ is[5]

$$\Psi(x,\theta,\lambda) = C(x,\theta) + \lambda^\alpha A_\alpha(x,\theta) + (\lambda\gamma^{mnpqr}\lambda)A^*_{mnpqr}(x,\theta) + \\ + \lambda^\alpha\lambda^\beta\lambda^\gamma C^*_{\alpha\beta\gamma}(x,\theta) + \cdots, \tag{5.8}$$

where ... includes superfields with more than three powers of λ^α. Since $Q\Psi = -i\lambda^\alpha D_\alpha C - i\lambda^\alpha\lambda^\beta D_\alpha A_\beta + \cdots$, $Q\Psi = 0$ implies that $A_\alpha(x,\theta)$ satisfies the equation of motion $\lambda^\alpha\lambda^\beta D_\alpha A_\beta = 0$. But since $\lambda^\alpha\lambda^\beta$ is proportional to $(\lambda\gamma^{mnpqr}\lambda)\gamma^{\alpha\beta}_{mnpqr}$, this implies that $D\gamma^{mnpqr}A = 0$, which is the linearized version of the super-Yang-Mills equation of motion of (4.3). Furthermore, if one defines the gauge parameter $\Omega = i\Lambda + \lambda^\alpha\omega_\alpha + \cdots$, the gauge transformation $\delta\Psi = Q\Omega$ implies $\delta A_\alpha = D_\alpha\Lambda$ which is the linearized super-Yang-Mills gauge transformation of (4.4).

As was shown in [10], the only states at non-zero momentum in the cohomology of Q are the on-shell super-Yang-Mills gluon and gluino, $a_m(x)$ and $\chi^\alpha(x)$, and their antifields, $a^{*m}(x)$ and $\chi^*_\alpha(x)$.[6] Since gauge invariances of the antifields correspond to equations of motion of the fields and vice versa, one expects a^{*m} and χ^*_α to satisfy the linearized equations of motion $\partial_m a^{*m} = 0$ with the linearized gauge invariances

$$\delta a^{*m} = \partial_n(\partial^n s^m - \partial^m s^n), \qquad \delta\chi^*_\alpha = \gamma^m_{\alpha\beta}\partial_m\kappa^\beta, \tag{5.9}$$

where s^m and κ^β are gauge parameters.

The fields a_m and χ^α appear in components of A_α as in (4.7), and the antifields a^{*m} and χ^*_α appear in components of the ghost-number +2 superfield A^*_{mnpqr} of (5.8). Using $Q\Psi = 0$ and $\delta\Psi = Q\Omega$, A^*_{mnpqr} satisfies the linearized equation of motion $\lambda^\alpha(\lambda\gamma^{mnpqr}\lambda)D_\alpha A^*_{mnpqr} = 0$ with the linearized gauge invariance $\delta A^*_{mnpqr} = \gamma^{\alpha\beta}_{mnpqr}D_\alpha\omega_\beta$. Expanding ω_α and A^*_{mnpqr} in components, one learns that A^*_{mnpqr} can be gauged to the form

$$A^*_{mnpqr} = \left(\theta\gamma_{[mnp}\theta\right)\left(\theta\gamma_{qr]}\right)^\alpha\chi^*_\alpha(x) + \left(\theta\gamma_{[mnp}\theta\right)\left(\theta\gamma_{qr]s}\theta\right)a^{*s}(x) + \cdots, \tag{5.10}$$

[5]It will be assumed that Ψ transforms covariantly under Lorentz transformations, which implies that it only depends on γ and u_{ab} through the combination λ^α of (5.3).

[6]The presence of the antifields can be seen from the doubling of the cohomology at ghost-numbers +1 and +2.

where χ^*_α and a^{*s} satisfy the equations of motion and residual gauge invariances of (5.9), and … involves terms higher order in θ^α which depend on derivatives of χ^*_α and a^{*s}.

As will be shown in the appendix, there are also zero momentum states in the cohomology of Q. In addition to the states described by the zero-momentum gluon, gluino, antigluon, and antigluino, there are also zero-momentum ghost and antighost states c and c^* in the $\theta = 0$ component of the ghost-number zero superfield, $C(x,\theta) = c(x) + \cdots$, and in the $(\theta)^5$ component of the ghost-number $+3$ superfield, $C^*_{\alpha\beta\gamma}(x,\theta) = \cdots + c^*(x)(\gamma^m\theta)_\alpha(\gamma^n\theta)_\beta(\gamma^p\theta)_\gamma(\theta\gamma_{mnp}\theta) + \cdots$. So although Ψ of (5.8) contains superfields of arbitrarily high ghost number, only superfields with ghost-number between zero and three contain states in the cohomology of Q.

The linearized equations of motion and gauge invariances $Q\Psi = 0$ and $\delta\Psi = Q\Omega$ are easily generalized to the non-linear equations of motion and gauge invariances

$$Q\Psi + g\Psi\Psi = 0\,, \qquad \delta\Psi = Q\Omega + g[\Psi,\Omega]\,, \tag{5.11}$$

where Ψ and Ω transform in the adjoint representation of the gauge group. For the superfield $A_\alpha(x,\theta)$, (5.11) implies the super-Yang-Mills equations of motion and gauge transformations of (4.3) and (4.4). Furthermore, the equations of motion and gauge transformation of (5.11) can be obtained from the spacetime action[7]

$$\mathcal{S} = \mathrm{Tr}\int d^{10}x \left\langle \frac{1}{2}\Psi Q\Psi + \frac{g}{3}\Psi\Psi\Psi \right\rangle \tag{5.12}$$

using the normalization definition that

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1\,. \tag{5.13}$$

Although (5.13) may seem strange, it resembles the normalization of (3.7) in that $\langle\Psi\rangle = c^*(x)$ where $c^*(x)$ is the spacetime antighost.[8] After writing (5.12) in terms of

[7]This spacetime action was first proposed to me by John Schwarz and Edward Witten. Because the action only involves integration over five θ's, it is not manifestly spacetime supersymmetric. Nevertheless, the equations of motion coming from this action have the same physical content as the manifestly spacetime supersymmetric equations of motion $Q\Psi + g\Psi\Psi = 0$. This is because all components in $Q\Psi + g\Psi\Psi = 0$ with more than five θ's are auxiliary equations of motion. So removing these equations of motion only changes auxiliary fields to gauge fields but does not affect the physical content of the theory. Unfortunately, this does not seem to be true after including the massive modes of the superstring. So there does not appear to exist a cubic superstring field theory action which reproduces the equations of motion $Q\Psi + g\Psi\times\Psi = 0$ where Ψ is the superstring field and $\times$ is Witten's midpoint interaction.

[8]It would be interesting to try to derive (5.11) from an eleven-dimensional action, in analogy to the four-dimensional Chern-Simons action of footnote 3. One natural guess would be to extend λ and θ to eleven-dimensional spinors and define the non-vanishing normalization as $\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\lambda\gamma^q\theta)(\theta\gamma_{mnpq}\theta)\rangle = 1$.

component fields and integrating out auxiliary fields, it should be possible to show that (5.12) reduces to the standard Batalin-Vilovisky action for super-Yang-Mills,

$$\mathcal{S} = \mathrm{Tr} \int d^{10}x \left(\frac{1}{4} f_{mn} f^{mn} + \chi^\alpha \gamma^m_{\alpha\beta} \left(\partial_m \chi^\beta + ig \left[a_m, \chi^\beta \right] \right) + \right.$$
$$\left. + i a^{*m} (\partial_m c + ig [a_m, c]) - g \chi^*_\alpha \{ \chi^\alpha, c \} - g c c c^* \right) . \tag{5.14}$$

5.3 Non-abelian super-Yang-Mills background

As in Chern-Simons theory, one can modify the worldline action of (5.6) to describe the superparticle in a non-abelian super-Yang-Mills background with SO(N) (or U(N)) gauge group by including N real (or complex) fermions. For SO(N) gauge group, the worldline action is

$$S = \int d\tau \left(\dot{x}^m P_m - \frac{1}{2} P^m P_m + \dot{\theta}^\alpha p_\alpha - \frac{i}{2} \eta_I \nabla \eta_I + \frac{1}{2} \dot{u}_{ab} v^{ab} + \dot{\gamma} \beta \right), \tag{5.15}$$

where

$$\nabla \eta_I = \dot{\eta}_I + g \eta_J \left(\dot{\theta}^\alpha \tilde{A}^{IJ}_\alpha + \Pi^m \tilde{B}^{IJ}_m + d_\alpha \tilde{W}^{\alpha\ IJ} + \frac{1}{2} N^{mn} \tilde{F}^{IJ}_{mn} \right), \tag{5.16}$$

$\tilde{A}^{IJ}_\alpha$ is the background super-Yang-Mills gauge field, $\tilde{B}^{IJ}_m$, $\tilde{W}^{\alpha\ IJ}$ and $\tilde{F}^{IJ}_{mn}$ are background superfields constructed from $\tilde{A}^{IJ}_\alpha$ as in (4.5), Π^m is defined in (2.2), and N_{mn} is defined in (5.7).

As will now be shown, the coupling of $\tilde{A}^{IJ}_\alpha$ has been chosen such that $\lambda^\alpha d_\alpha$ is conserved. To show this, note that $\dot{\lambda}^\alpha = -\frac{ig}{8} \eta_I \eta_J (\gamma^{mn} \lambda)^\alpha F^{IJ}_{mn}$ and

$$\begin{aligned}
\dot{d}_\alpha &= \frac{ig}{2} \frac{\partial}{\partial \tau} \left(\eta_I \eta_J \tilde{A}^{IJ}_\alpha \right) - \frac{ig}{2} \eta_I \eta_J \times \\
&\quad \times \left(-\dot{\theta}^\beta D_\alpha \tilde{A}^{IJ}_\beta + \Pi^m D_\alpha \tilde{B}^{IJ}_m - d_\beta D_\alpha \tilde{W}^{\beta\ IJ} + \right. \\
&\qquad \left. + \frac{1}{2} N_{mn} D_\alpha \tilde{F}^{IJ}_{mn} + \gamma^m_{\alpha\beta} \left(i \dot{\theta}^\beta \tilde{B}^{IJ}_m + \Pi_m \tilde{W}^{\beta\ IJ} \right) \right) \\
&= \frac{ig}{2} \eta_I \eta_J \left(\dot{\theta}^\beta \left(D_{(\alpha} \tilde{A}^{IJ}_{\beta)} + g \tilde{A}^{IK}_{(\alpha} \tilde{A}^{KJ}_{\beta)} - i \gamma^m_{\alpha\beta} \tilde{B}^{IJ}_m \right) + \right. \\
&\qquad + \Pi^m \left(- \left(\nabla_\alpha \tilde{B}_m \right)^{IJ} + \partial_m \tilde{A}^{IJ}_\alpha + \gamma_{m\ \alpha\beta} W^{\beta\ IJ} \right) + \\
&\qquad \left. + d_\beta \left(\nabla_\alpha \tilde{W}^\beta \right)^{IJ} - \frac{1}{2} N^{mn} \left(\nabla_\alpha \tilde{F}_{mn} \right)^{IJ} \right),
\end{aligned} \tag{5.17}$$

where $(\nabla_\alpha s)^{IJ} = D_\alpha s^{IJ} + g(s^{IK} \tilde{A}^{KJ}_\alpha - s^{JK} \tilde{A}^{KI}_\alpha)$ and the equations of motion $\dot{\theta}^\alpha = -\frac{ig}{2} \eta_I \eta_J \tilde{W}^{\alpha\ IJ}$ and $P^m = \Pi^m$ have been used. So using the definitions of (4.5),

$$\frac{\partial}{\partial \tau} (\lambda^\alpha d_\alpha) = -\frac{ig}{4} \eta_I \eta_J \lambda^\alpha N^{mn} \left(\nabla_\alpha \tilde{F}_{mn} \right)^{IJ} . \tag{5.18}$$

But F^{IJ}_{mn} satisfies $(\nabla_\alpha \tilde{F}_{mn})^{IJ} = \frac{i}{8}\gamma_{\alpha\beta\ [m}(\nabla_{n]}W^\beta)^{IJ}$ where $(\nabla_n s)^{IJ} = D_n s^{IJ} + g(s^{IK}\tilde{B}^{KJ}_n - s^{JK}\tilde{B}^{KI}_n)$. Also, one can check that $N^{mn} = \frac{1}{2}\lambda\gamma^{mn}w$ where w_α is an anti-Weyl spinor with SU(5) components

$$w_+ = \beta\,, \qquad w^{ab} = v^{ab}\,, \qquad w_a = 0\,. \tag{5.19}$$

So

$$\begin{aligned}\lambda^\alpha N^{mn}\left(\nabla_\alpha \tilde{F}_{mn}\right)^{IJ} &= \frac{i}{16}\lambda^\alpha(\lambda\gamma^{mn}w)\gamma_{\alpha\beta\ [m}\left(\nabla_{n]}\tilde{W}^\beta\right)^{IJ} \\ &= -\frac{i}{8}\left(\lambda^\delta w_\delta\right)\lambda^\alpha\gamma^n_{\alpha\beta}\left(\nabla_n\tilde{W}^\beta\right)^{IJ}\end{aligned} \tag{5.20}$$

using (5.2) and the gamma-matrix identity that $\eta_{mn}\gamma^m_{(\alpha\beta}\gamma^n_{\delta)\kappa} = 0$. But one now can use that $\gamma^n_{\alpha\beta}(\nabla_n\tilde{W}^\beta)^{IJ} = 0$ to imply that (5.20) vanishes, and therefore $\lambda^\alpha d_\alpha$ is conserved.

Finally, it will be shown that $Q(\eta_I\eta_J\Psi^{IJ}) = 0$ gives the correct equation of motion in the presence of the background gauge field $\tilde{A}^{IJ}_\alpha$. Since the canonical momentum for θ^α and x^m are

$$\begin{aligned}\frac{\partial L}{\partial\dot{\theta}^\alpha} &= p_\alpha - \frac{ig}{2}\eta_I\eta_J\left(\tilde{A}^{IJ}_\alpha - \frac{i}{2}\theta^\beta\gamma^m_{\alpha\beta}\tilde{B}^{IJ}_m\right)\,, \\ \frac{\partial L}{\partial\dot{x}^m} &= P_m - \frac{ig}{2}\eta_I\eta_J\tilde{B}^{IJ}_m\,,\end{aligned} \tag{5.21}$$

one finds that

$$\begin{aligned}Q = \lambda^\alpha d_\alpha &= \lambda^\alpha\left(p_\alpha + \frac{i}{2}P_m\gamma^m_{\alpha\beta}\theta^\beta\right) \\ &= \lambda^\alpha\left(\frac{\partial L}{\partial\dot{\theta}^\alpha} + \frac{i}{2}\frac{\partial L}{\partial\dot{x}^m}\gamma^m_{\alpha\beta}\theta^\beta + \frac{ig}{2}\eta_I\eta_J\tilde{A}^{IJ}_\alpha\right).\end{aligned} \tag{5.22}$$

So

$$\begin{aligned}Q(\eta_I\eta_J\Psi^{IJ}) = -i\eta_I\eta_J\Big(&\lambda^\alpha(\nabla_\alpha C)^{IJ} + \lambda^\alpha\lambda^\beta(\nabla_\alpha A_\beta)^{IJ} + \\ &+ \lambda^\alpha\left(\lambda\gamma^{mnpqr}\lambda\right)\left(\nabla_\alpha A^*_{mnpqr}\right)^{IJ} + \cdots\Big)\,,\end{aligned} \tag{5.23}$$

where $(\nabla_\alpha s)^{IJ} = D_\alpha s^{IJ} + g(s^{IK}\tilde{A}^{KJ}_\alpha - s^{JK}\tilde{A}^{KI}_\alpha)$, which correctly covariantizes the supersymmetric derivatives with respect to the background gauge field.

A. Superparticle cohomology at zero momentum

In this appendix, the zero momentum cohomology of $Q = \lambda^\alpha d_\alpha$ will be computed for arbitrary ghost number and shown to correspond to the ghost, gluon, gluino, antigluino, antigluon, and antighost of super-Yang-Mills. Since $Q = \lambda^\alpha p_\alpha$ when

$P_m = 0$, the only reason one has non-trivial cohomology is because λ^α is constrained by (5.2). It will now be proven that the cohomology of $Q = \lambda^\alpha p_\alpha$ with constrained λ^α is equivalent to the "linear" cohomology of $\widehat{Q}$ with unconstrained λ^α where

$$\begin{aligned}\widehat{Q} = {}& \lambda^\alpha p_\alpha + (\lambda\gamma^m\lambda)b_m + c^m(\lambda\gamma_m f) + (\lambda\gamma_m\lambda)(j\gamma^m g) - 2(j_\alpha\lambda^\alpha)(g_\beta\lambda^\beta) + \\ & + (k\gamma_m\lambda)r^m + (\lambda\gamma^m\lambda)s_m t\,, \end{aligned} \tag{A.1}$$

and "linear" cohomology signifies elements in the cohomology of $\widehat{Q}$ which are at most linearly dependent on the new variables $(c^m, g_\alpha, k^\alpha, s_m, u)$. Note that (c^m, b_m), (g_α, f^α), (k^α, j_α), (s_m, r^m), and (u, t) are pairs of new variables and their conjugate momentum which have been added to the Hilbert space. The pairs (c^m, b_m), (k^α, j_α) and (u, t) are fermions of ghost-number $(1, -1)$, $(2, -2)$ and $(3, -3)$, respectively, and the pairs (g_α, f^α) and (s_m, r^m) are bosons of ghost-number $(1, -1)$ and $(2, -2)$, respectively.

A.1 Equivalence of Q and $\widehat{Q}$ cohomologies

To relate the cohomologies of Q and $\widehat{Q}$, consider a state $F(\lambda, \theta)$ in the cohomology of Q with constrained λ^α. Then $QF = (\lambda\gamma^m\lambda)\tau_m$ for some τ_m. So $\widehat{Q}(F - c^m\tau_m) = c^m Q\tau_m$. But $Q^2 F = 0$ implies that $(\lambda\gamma^m\lambda)Q\tau_m = 0$, which implies that $Q\tau_m = \lambda\gamma_m\psi$ for some ψ^α. So $\widehat{Q}(F - c^m\tau_m - g_\alpha\psi^\alpha) = -g_\alpha Q\psi^\alpha$. But $Q^2\tau_m = 0$ implies that $\lambda\gamma_m Q\psi = 0$, which implies that $Q\psi^\alpha = (\lambda\gamma^n\lambda)(\gamma_n\rho)^\alpha - 2\lambda^\alpha(\lambda^\beta\rho_\beta)$ for some ρ_β. This line of argument continues until one has $\widehat{Q}(F - c^m\tau_m - g_\alpha\psi^\alpha + k^\alpha\rho_\alpha + s_m\sigma^m + u\kappa) = uQ\kappa$. Finally, $Q^2\sigma^m = 0$ implies that $(\lambda\gamma^m\lambda)Q\kappa = 0$, which implies that $Q\kappa = 0$. So for any state $F(\lambda, \theta)$ in the cohomology of Q with constrained λ^α, one can construct a state $\widehat{F}$ annihilated by $\widehat{Q}$ which is at most linear in $(c^m, g_\alpha, k^\alpha, s_m, u)$.

To show that $\widehat{F}$ is in the cohomology of $\widehat{Q}$, suppose that $\widehat{F} = \widehat{Q}\widehat{\Omega}$ for some $\widehat{\Omega} = \Omega(\lambda, \theta) + c^m\xi_m(\lambda, \theta) + \cdots$. Then since F is the term in $\widehat{F}$ which is independent of the new variables, $F = Q\Omega + (\lambda\gamma^m\lambda)\xi_m$. But this is not possible if F is in the cohomology of Q with constrained λ^α, so $\widehat{F} \neq \widehat{Q}\widehat{\Omega}$ for any $\widehat{\Omega}$.

Now suppose that one starts with a state $\widehat{F}$ in the cohomology of $\widehat{Q}$ which is at most linear in $(c^m, g_\alpha, k^\alpha, s_m, u)$, i.e.

$$\widehat{F} = F(\lambda,\theta) + c^m\tau_m(\lambda,\theta) + g_\alpha\psi^\alpha(\lambda,\theta) + k^\alpha\rho_\alpha(\lambda,\theta) + s_m\sigma^m(\lambda,\theta) + u\kappa(\lambda,\theta)\,. \tag{A.2}$$

Then $\widehat{Q}\widehat{F} = 0$ implies that $QF = -(\lambda\gamma^m\lambda)\tau_m$, so F is annihilated by Q with constrained λ^α.

To show that F is in the cohomology of Q, suppose that $F = Q\Omega + (\lambda\gamma^m\lambda)\xi_m$ for some $\Omega(\lambda, \theta)$ and $\xi_m(\lambda, \theta)$. Then $\widehat{F} = \widehat{Q}(\Omega + c^m\xi_m) + c^m(\tau_m + Q\xi_m) + g_\alpha\psi^\alpha + \cdots$. So $\widehat{Q}\widehat{F} = 0$ implies that $(\lambda\gamma^m\lambda)(\tau_m + Q\xi_m) = 0$, which implies that $\tau_m + Q\xi_m = \lambda\gamma^m\chi$ for some χ^α. So $\widehat{F} = \widehat{Q}(\Omega + c^m\xi_m + g_\alpha\chi^\alpha) + g_\alpha(\psi^\alpha - Q\chi^\alpha) + k^\alpha\rho_\alpha + \cdots$. This argument continues until one finds that $\widehat{F} = \widehat{Q}(\Omega + c^m\xi_m + g_\alpha\chi^\alpha + \cdots + u\varepsilon)$, which is not possible if $\widehat{F}$ is in the cohomology of $\widehat{Q}$. So it has been proven that the cohomology of Q with constrained λ^α is equivalent to the "linear" cohomology of $\widehat{Q}$ with unconstrained λ^α.

A.2 Evaluation of $\widehat{Q}$ cohomology

Since $\widehat{Q} = \lambda^\alpha \widehat{p}_\alpha$ where λ^α is unconstrained and

$$\widehat{p}_\alpha = p_\alpha + (\gamma^m\lambda)_\alpha b_m + c^m(\gamma_m f)_\alpha + (\gamma_m\lambda)_\alpha(j\gamma^m g) - 2j_\alpha(g_\beta\lambda^\beta) + (\gamma_m k)_\alpha r^m + (\gamma^m\lambda)_\alpha s_m t\,,$$

it is easy to evaluate the cohomology of $\widehat{Q}$. Using the quartet mechanism, one can choose a gauge such that states in the cohomology are independent of λ^α and θ^α. So states in the "linear" cohomology are represented by the elements $(1, c^m, g_\alpha, k^\alpha, s_m, u)$, which have ghost-number $(0,1,1,2,2,3)$, respectively.

To relate these elements to states in the cohomology of Q, one needs to find gauge-invariant version of these elements which commute with $\widehat{Q}$. For example, $c^m - i\lambda\gamma^m\theta$ and $g_\alpha - i(\theta\gamma_m)_\alpha c^m + \frac{2}{3}(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha$ commute with $\widehat{Q}$, so $\lambda\gamma^m\theta$ and $(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha$ are the states in the cohomology of Q which are associated with c^m and g_α. Similarly, one can show that $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\theta\gamma_{mn})^\alpha$ is the state associated with k^α, $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\theta\gamma_{mnp}\theta)$ is the state associated with s_p, and $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$ is the state associated with u.

Comparing these states with the superfields in (5.8), one finds that the zero momentum states in the cohomology of Q correspond to the ghost, gluon, gluino, antigluino, antigluon, and antighost of super-Yang-Mills.

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